Lattice path conjectures

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Abstract

This document introduces a number of conjectures related to lattice path enumeration and relationships with enumerating other structures.

Keywords: lattice paths; enumeration.

MSC: 05A15; 05C38; 05C12; 05C20; 05C30; 20M13; 05A10.

1 Introduction

The following document contains numerous conjectured relationships involving the enumeration of lattice paths using subsets of $\mathbb{Z}^2$ as step-sets. The author recently put out a paper with James East (see [1]) which begins to develop some theory on the topic of lattice path enumeration, not just analysing specific examples much like how lattice paths have traditionally been examined/explored, and contrast to what we shall be doing in this document.

When the number of lattice paths for a given step-set is equal to the number of objects for another structure, let X and Y be the computationally/algorithmically dearer and cheaper algorithms respectively for enumerating the lattice paths and enumerating the other objects. Any algorithm that uses X may be made more efficient by instead using Y.

There was not anything that special mathematically to finding the conjectures contained within (that is an attitude from someone at the research level, it would be very impressive from people at most other levels depending on what audience ends up reading this), just coming up with example step-sets and doing a bit of relatively simple programming to spout some numbers out.

I was going to clean it up a bit but am rather busy with other stuff at the moment, it would be interesting to see what different people are able to come up with in regards to explaining it to other people at lower/higher levels (maybe even writing up their own version of what the conjectures are, and what approaches people might take when playing with it, including trying to prove some of the conjectures or identify more interesting conjectures), which arguments people are able to come up with to prove the various conjectures, what applications may arise for example with bijections between lattice paths and other objects, so on and so forth, maybe even trying to create some kind of dictionary/encyclopaedia for different aspects (eg. step-sets, sequences that arise, etc.), discussing what the correct term is would be quite useful as well.

Enumerating the lattice paths/walks using Euclid's orchard as a step-set is kind of interesting too, it would be awesome if someone managed to find a connection to any sequence arising from enumerating paths/walks using a step-set that is a subset of Euclid's orchard (or a generalisation of it, including generalising the number of dimensions, or possibly even on the rational lattice $\mathbb{Q}^n$ rather than integer lattice $\mathbb{Z}^n$).

2 Powers of 2

Lattice paths using step-set $\{(a, 0) : a \in \mathbb{Z}^+\}$.

sequence A000079 on the OEIS.

3 k-Generalised Fibonacci numbers

Lattice paths using step-set $\{(a, 0) : a \in \{1, ..., k\}\}$ where $k \in \mathbb{Z}_{>0}$.

(sequence A092921 on the OEIS).

4 binomial$((m+1)n, n)$

Lattice paths using step-set $\{(1,1),(0,-m)\}$ where $m \in \mathbb{Z}_{>0}$.

$m = 1$ A000984

$m = 2$ A005809

$m = 3$ A005810

5 Expansion of $(1/2)*(1/(x+1)+1/(\sqrt{-3*x^2-2*x+1}))$

Lattice paths using step-set $\{(1,1),(1,-a) : a \in \mathbb{Z}_{>0}\}$.

A246437

6 G.f. satisfies: $A(x) = 1 + x * A(x)/(1 - x^2 * A(x)^2)$

These are the (ordered) odd trees. There is example input for the algorithms from [1] (there is an implementation of these algorithms at [2]) in the *example_input* directory for the repository containing this document.

Lattice paths that do not step below x-axis using step-set $\{(1,1),(1,-2k) : k \in \mathbb{Z}_{\geq 0}\}$.

A101785

Paths along $y = x - 2$ are maybe A000217 (though there's a few other matches up to 91 where I can check before latex breaks).

Paths along $y = x - 3$ are maybe A003600/A283551.

Column sums appear to be A143017.

Lattice paths that do not step below the x-axis using step-set $\{(1,-1),(1,2k) : k \in \mathbb{Z}_{\geq 0}\}$.

Paths along $y = 0$ seem to give A101785.

7 Number of Dyck n-paths all of whose ascents have lengths equal to 1 (mod 3)

Lattice paths that do not step below x-axis using step-set $\{(1,1),(1,-3k) : k \in \mathbb{Z}_{\geq 0}\}$.

A212383

8 Number of Dyck n-paths all of whose ascents have lengths equal to 1 (mod 4)

Lattice paths that do not step below x-axis using step-set $\{(1,1),(1,-4k) : k \in \mathbb{Z}_{\geq 0}\}$.

A212384

9 Number of Dyck n-paths all of whose ascents have lengths equal to 1 (mod 5)

Lattice paths that do not step below x-axis using step-set $\{(1,1),(1,-5k) : k \in \mathbb{Z}_{\geq 0}\}$.

A212385

10 Number of Dyck n-paths all of whose ascents have lengths equal to 1 (mod 6)

Lattice paths that do not step below x-axis using step-set $\{(1,1),(1,-6k) : k \in \mathbb{Z}_{\geq 0}\}$.

A212386

11 conjecture: Number of Dyck n-paths all of whose ascents have lengths equal to 1 (mod m)

Lattice paths that do not step below x-axis using step-set $\{(1,1),(1,-mk) : k \in \mathbb{Z}_{\geq 0}\}$.

12 binomial$(2n+1, n+1)$

Lattice paths using step-set $\{(1,1),(1,-k+1) : k \in \mathbb{Z}_{>0}\}$.

A001700

13 Number of compositions (= ordered integer partitions) of n into 2n parts

Lattice paths using step-set $\{(1,1),(1,-2k+1) : k \in \mathbb{Z}_{>0}\}$

A165817

Surprising that this isn't binomial(3n-1, n)? (based on the next two)

14 binomial(3n,n)/(2n+1) (enumerates ternary trees and also noncrossing trees)

Lattice paths using step-set $\{(1,-1),(1,2k-1) : k \in \mathbb{Z}_{>0}\}$

A001764 paths along $y=0$ and $y=1$.

paths along $x=3$ might be A000217 (can only check up to 45).

paths along $x=4$ might be A062748.

paths along $x=5$ might be A005718.

paths along $x=6$ might be A062749.

paths along $y=x-1$ might be A127927.

Lattice paths that do not step below x-axis using step-set $\{(1,1),(0,-2)\}$.

A001764 paths along $y=0$.

15 binomial$(4n-1, n)$

Lattice paths using step-set $\{(1,1),(1,-3k+1) : k \in \mathbb{Z}_{>0}\}$

A262977

16 binomial$(5n-1, n)$

Lattice paths using step-set $\{(1,1),(1,-4k+1) : k \in \mathbb{Z}_{>0}\}$

A163455

17 Pfaff-Fuss-Catalan sequence

Lattice paths that do not step below x-axis using step-set $\{(1,1),(0,-m)\}$ where $m \in \mathbb{Z}_{>0}$.

Or lattice paths that do not step below x-axis using step-set $\{(1,1),(1,-mk+1) : k \in \mathbb{Z}_{>0}\}$

$m=1$ A000108

$m=2$ A001764 **Table 1**.

$m=3$ A002293 **Table 2**.

$m=4$ A002294 **Table 3**.

A002295

A002296

A007556

A062994/A059967

A062744

A230388
A062993? Pfaff-Fuss-Catalan triangle?

18 Dissections of a convex polygon by non-intersecting diagonals into polygons with an even number of sides and having a total number of n edges (sides and diagonals)

Lattice paths that do not step below the x-axis using step-set $\{(1,1),(1,-2k) : k \in \mathbb{Z}_{>0}\}$.

A067955 **Table** 4

Paths along $y = x - 6$ appear to be A055998/A027379/A167544 or A141214 (I haven't checked up to where they differ).

Paths along $y = x - 7$ appear to be A028563.

Lattice paths that do not step below the x-axis using step-set $\{(1,-1),(1,2k) : k \in \mathbb{Z}_{>0}\}$.

Paths along $y = 0$, $y = 1$ and $y = 3$ seem to give A067955.

19 Even trees

A049124

Lattice paths that do not step below the x-axis using step-set $\{(1,1),(0,-2k+1) : k \in \mathbb{Z}_{>0}\}$.

Table 5.

Lattice paths that do not step below the x-axis using step-set $\{(1,-1),(0,2k-1) : k \in \mathbb{Z}_{>0}\}$.

paths along $y = 0$ and $y = 1$ seem to give A049124.

paths along $x = 0$ trivially give A000045.

paths along $x = 1$ seems to give A029907.

20 Motzkin sums, aka Riordan numbers

A097609, A005043

Lattice paths that do not cross below the x-axis using step-set $\{(1,1),(1,-k) : k \in \mathbb{Z}_{>0}\}$. Gives Motzkin paths of length n with k horizontal steps at level 0.

Table 6

Also lattice paths that do not cross below the x-axis using step-set $\{(1,k),(1,-1) : k \in \mathbb{Z}_{>0}\}$.

Table 7

Row $y = 2$ in **Table** 7 gives A002026 (generalised ballot numbers, first differences of Motzkin sums).

Column $x = 3$ in **Table** 7 gives A022856/A152948/A089071/A133263/A238531 (see OEIS pages).

21 Schroeder's second problem (generalized parentheses); also called super-Catalan numbers or little Schroeder numbers

Lattice paths that do not cross below x-axis using step-set $\{(1,1),(0,-z) : z \in \mathbb{Z}_{>0}\}$.

A001003 **Table** 8

22 Number of blobs with 2n+1 edges

Lattice paths that do not cross below x-axis using step-set $\{(1,1),(0,-2z) : z \in \mathbb{Z}_{>0}\}$.

$m = 2$ A003168 **Table** 9

Lattice paths that do not step below the x-axis using step-set $\{(1,-1),(0,2k) : k \in \mathbb{Z}_{>0}\}$.

Paths along $y = 0$, $y = 1$ and $y = 2$ seem to give A003168.

Column $x = 1$ might give A001792 (have only checked up to 576 and there's still other options).

Column $x = 2$ might give A049611/A084851

23 Sylvester classes of m-packed words

Lattice paths that do not cross below the x-axis using step-set $\{(1,1),(0,-mz):z\in\mathbb{Z}_{>0}\}$ where $m\geq 3$.

$m=3$ A243659 **Table 10**

A243667

A243668

24 Factorials

Lattice paths that do not cross above $y=x$ or below $y=-x$ using the step-set

$S=\{(1,2a-1),(1,-2a+1):a\in\mathbb{Z}_{>0}\}$.

Paths along $y=x+a$ where $a\in\mathbb{Z}$ give the factorials (A000142).

Column sums are trivially equal to the factorials.

Paths along $y=2a$ where $a\in\mathbb{Z}$ give $(2n)!$ (A010050).

Paths along $y=2a+1$ where $a\in\mathbb{Z}$ give $(2n+1)!$ (A009445).

Lattice paths that do not cross above $y=x$ or below the x-axis using the step-set $S=\{(1,a):a\in\mathbb{Z}\}$.

Paths along $y=x$ where $a\in\mathbb{Z}_{>0}$ give factorials (A000142). Also paths along $y=a$ where $a\in\mathbb{Z}_{>0}$

Column sums are trivially also equal to the factorials.

Lattice paths that do not cross above $y=x$ or below the x-axis using the step-set $S=\{(1,a),(1,-a):a\in\mathbb{Z}_{>0}\}$. (SEE ALSO ALTERNATING FACTORIALS).

Paths along $y=x$ (from $(0,0)$ to $(n+1,n+1)$ give factorials (A000142).

25 Alternating factorials

Lattice paths that do not cross above $y=x$ or below the x-axis using the step-set $S=\{(1,a),(1,-a):a\in\mathbb{Z}_{>0}\}$.

Paths along x-axis (from $(0,0)$ to $(0,n+1)$) give alternating factorials (A005165) (also the paths along $y=2$).

Paths along $y=x$ (from $(0,0)$ to $(n+1,n+1)$ give factorials (A000142).

Column sums also give factorials (trivially the same as above).

Paths along $y=1$ give A153229/A058006 (also the paths along $y=3$).

Paths along $y=x-1$ give A001563/A094258/A094304

26 Double factorial of even numbers

Lattice paths that do not cross above $y=x$ or below the $y=-x$ using the step-set $S=\{(1,a),(1,-a):a\in\mathbb{Z}_{>0}\}$.

Paths along $y=x$ give the double factorial of even numbers (A000165) (also the paths along $y=-x$).

Column sums also trivially give double factorial of even numbers.

Paths along $y=x-1$ give A014481 (also paths along $y=1-x$).

Paths along $y=1$ give A290044 (also paths along $y=-1$).

27 $(n!)^2$ and $n!(n+1)!$

Lattice paths that do not cross above $y=x$ or below $y=0$ using the step set

$S=\{(1,2a-1),(1,-2a+1):a\in\mathbb{Z}_{>0}\}$.

Paths along $y=2a$ where $a\in\mathbb{Z}_{\geq 0}$ give A001044 $(n!)^2$.

Paths along $y=2a+1$ where $a\in\mathbb{Z}_{\geq 0}$ give A010790 $n!(n+1)!$.

Paths along $y=x-a$ where $a\in\mathbb{Z}_{\geq 0}$ give A001044 with even index and A010790 with odd index. With the combined sequence appearing to be A010551 though this relationship to A001044/A010790 is not mentioned on OEIS.

Column sums trivially give A010551.

A010551 is also given by lattice paths that do not cross above $y=x$ or below $y=0$ using the step set $S=\{(1,2a):a\in\mathbb{Z}\}$.

Paths along $y=a$ where $a\in\mathbb{Z}$ give A010551.

Column sums also trivially give A010551.

28 Number of permutations in the symmetric group S_n that have odd order

Lattice paths that do not cross above $y = x$ or below $y = -x$ using the step set $S = \{(1, 2a) : a \in \mathbb{Z}\}$.
Paths along $y = a$ where $a \in \mathbb{Z}$ give A000246.
Column sums also trivially give A000246.

29 Number of n X 1 arrays of permutations of 0..n*1-1 with rows nondecreasing modulo 2 and columns nondecreasing modulo 3

Lattice paths that do not cross above $y = x$ or below $y = 0$ using the step set $\{(1, 3a) : a \in \mathbb{Z}\}$.
Paths along $y = a$ where $a \in \mathbb{Z}_{\geq 0}$ give A264557.
Column sums also trivially give A264557.

30 Number of n X 1 arrays of permutations of 0..n*1-1 with rows nondecreasing modulo 2 and columns nondecreasing modulo 4

Lattice paths that do not cross above $y = x$ or below $y = 0$ using the step set $\{(1, 4a) : a \in \mathbb{Z}\}$.
Paths along $y = a$ where $a \in \mathbb{Z}_{\geq 0}$ give A264635.
Column sums also trivially give A264635.

31 Number of n X 1 arrays of permutations of 0..n*1-1 with rows nondecreasing modulo 2 and columns nondecreasing modulo 5

Lattice paths that do not cross above $y = x$ or below $y = 0$ using the step set $\{(1, 5a) : a \in \mathbb{Z}\}$.
Paths along $y = a$ where $a \in \mathbb{Z}_{\geq 0}$ give A264656.
Column sums also trivially give A264656.

32 Number of n X 1 arrays of permutations of 0..n-1 with rows nondecreasing modulo 2 and columns nondecreasing modulo 6

Lattice paths that do not cross above $y = x$ or below $y = 0$ using the step set $\{(1, 6a) : a \in \mathbb{Z}\}$.
Paths along $y = a$ where $a \in \mathbb{Z}_{\geq 0}$ give A264701.
Column sums also trivially give A264701.

33 step set $\{(a, 1) : a \in \mathbb{Z}_{\geq 0}\}$

Lattice paths using the step set $\{(a, 1) : a \in \mathbb{Z}_{\geq 0}\}$ (or $\{(a, 1) : a \in \mathbb{Z}_{>0}\}$).
Paths along $y = 3$ and $x = 2$ give A000217.
Paths along $y = 4$ and $x = 3$ give A000292.
Paths along $y = 5$ and $x = 4$ give A000332.
Paths along $y = 6$ and $x = 5$ give A000389.
Paths along $y = 7$ and $x = 6$ give A000579.
Paths along $y = 8$ and $x = 7$ give A000580.
Paths along $y = 9$ and $x = 8$ give A000581.
Paths along $y = 10$ and $x = 9$ give A000582.

34 step set $\{(a, \pm 1) : a \in \mathbb{Z}_{\geq 0}\}$

Lattice paths using the step set $\{(a, \pm 1) : a \in \mathbb{Z}_{>0}\}$.
Triangle of numbers formed seems to be A025177
along y=+-1 you get A025179.

along y=+-2 you get A025180.

along y=+-3 you get A025181.

presumably this continues

along y=x-2 and y=-x+2 you get (maybe) A000124 (there's still 12 results with the numbers I have generated)

along y=x-3 and y=-x+3 you get (maybe) A000292

along y=x-4 and y=-x+4 you get A086274

35 $a(n) = 2 * n^2 + 1$

Lattice paths using the step set $\{(a, \pm 1), (1, 0) : a \in \mathbb{Z}_{>0}\}$ or $\{(1, 0), (1, 1), (2, 1), (3, 1), (1, -1), (1, -2), (1, -3)\}$.

paths along $y = x - 2$ and $y = -x + 2$ give A058331.

With $\{(1, 0), (1, 1), (2, 1), (3, 1), (4, 1), (1, -1)\}$, paths along $y = x - 3$ might be $+\frac{4}{3}x^3 - 2x^2 + \frac{11}{3}x - 1$ according to OEIS.

36 Number of branches in all ordered trees with n edges

Lattice paths using the step set $\{(a, 0), (a, \pm 1) : a \in \mathbb{Z}_{>0}\}$.

Paths along x-axis give A051924.

Paths along $y = \pm 1$ give A076540.

Paths along $y = x - 2$ and $y = -x + 2$ give A084849.

37 step set $\{(0, 1), (a, 0), (a, \pm 1) : a \in \mathbb{Z}_{>0}\}$

Lattice paths using the step set $\{(0, 1), (a, 0), (a, \pm 1) : a \in \mathbb{Z}_{>0}\}$.

All numbers seem to give A054120, though I have a hard time giving the bijection to the triangle of numbers on OEIS.

Paths along x-axis give A054122.

Paths along $y = \pm 1$ give A052392.

Paths along $x = 3$ give A054121.

38 First differences of the central Delannoy numbers (A001850)

Lattice paths using the step set $\{(0, 1), (a, \pm 1) : a \in \mathbb{Z}_{>0}\}$.

Numbers along x-axis give A110170.

Numbers along $y = \pm 1$ give A104550.

Numbers along $x = 2$ give A005893.

39 step set $\{(0, 1), (a, 0), (a, 1) : a \in \mathbb{Z}_{>0}\}$

Lattice paths using the step set $\{(0, 1), (a, 0), (a, 1) : a \in \mathbb{Z}_{>0}\}$.

Numbers along $y = 1$ give A001792.

Numbers along $y = 2$ give A001793.

Numbers along $y = 3$ give A001794.

Numbers along $y = 4$ give A006974.

Numbers along $y = 5$ give A006975.

Numbers along $y = 6$ give A006976.

Numbers along $y = 7$ give A209404.

Numbers along $x = 2$ give A001105.

Numbers along $x = 3$ give A002492.

Numbers along $x = 4$ give A072819.

Numbers along $y = x + 2$ might be A275827.

Sums of $y = -x + a$ lines give a sequence, however I can't get enough terms to work out what.

40 step set $\{(1,a),(a,1) : a \in \mathbb{Z}_{>0}\}$

Lattice paths using the step set $\{(1,a),(a,1) : a \in \mathbb{Z}_{>0}\}$.

Numbers along $y = x$ might be A171155.

Numbers along $y = 3$ and $x = 3$ might be A022856/A089071 A152948/A133263.

All numbers seem to be A047080.

Lattice paths that do not step above $y = x$ using the step set $\{(1,a),(a,1) : a \in \mathbb{Z}_{>0}\}$.

Numbers along $y = x$ might be A082582/A086581/A025242.

Numbers along $y = 3$ might be A000124/A152947.

Numbers along $y = 4$ might be A177787.

41 Calculated values

In this section we list calculated values for the number of lattice paths for various step-sets considered.

14															1
13														1	
12													1		13
11												1		12	
10											1		11		88
9										1		10		75	
8									1		9		63		408
7								1		8		52		320	
6							1		7		42		245		1428
5						1		6		33		182		1020	
4					1		5		25		130		700		3876
3				1		4		18		88		455		2448	
2			1		3		12		55		273		1428		7752
1		1		2		7		30		143		728		3876	
0	1		1		3		12		55		273		1428		7752
y \ x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

Table 1: Number of $\{(1,1),(0,-2)\}$-paths that do not cross below the x-axis.

15																1
14															1	
13														1		
12													1			13
11												1			12	
10											1			11		
9										1			10			85
8									1			9			72	
7								1			8			60		
6							1			7			49			357
5						1			6			39			272	
4					1			5			30			200		
3				1			4			22			140			969
2			1			3			15			91			612	
1		1			2			9			52			340		
0	1			1			4			22			140			969
y \ x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Table 2: Number of $\{(1,1),(0,-3)\}$-paths that do not cross below the x-axis.

y \ x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
16																	1
15																1	
14															1		
13														1			
12													1				13
11												1				12	
10											1				11		
9										1				10			
8									1				9				81
7								1				8				68	
6							1				7				56		
5						1				6				45			
4					1				5				35				285
3				1				4				26				204	
2			1				3				18				136		
1		1				2				11				80			
0	1				1				5				35				285

Table 3: Number of $\{(1,1),(0,-4)\}$-paths that do not cross below the x-axis.

y \ x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
14															1
13														1	
12													1		
11												1			12
10											1			11	
9										1			10		10
8									1			9		9	63
7								1			8		8	52	8
6							1			7		7	42	7	98
5						1			6		6	33	6	78	188
4					1			5		5	25	5	60	135	105
3				1			4		4	18	4	44	92	78	368
2			1			3		3	12	3	30	58	54	237	357
1		1			2		2	7	2	18	32	33	134	195	366
0	1			1		1	3	1	8	13	15	56	79	157	399

Table 4: Number of $\{(1,1),(1,-2k) : k \in \mathbb{Z}_{>0}\}$-paths that do not cross below the x-axis.

y \ x	0	1	2	3	4	5	6	7	8	9	10	11
11												1
10											1	11
9										1	10	65
8									1	9	54	282
7								1	8	44	216	1014
6							1	7	35	161	721	3192
5						1	6	27	116	495	2112	9035
4					1	5	20	80	325	1336	5550	23275
3				1	4	14	52	201	796	3212	13160	54600
2			1	3	9	31	114	435	1706	6834	27840	114975
1		1	2	5	16	56	206	786	3082	12342	50260	207493
0	1	1	2	6	20	71	264	1015	4002	16094	65758	272208

Table 5: Number of $\{(1,1),(0,-2k+1) : k \in \mathbb{Z}_{>0}\}$-paths that do not cross below the x-axis.

References

[1] James East and Nicholas Ham. Lattice paths and submonoids of $\mathbb{Z}^2$. *Preprint*, 2015, arXiv:1811.05735.

14															1
13														1	
12													1		13
11												1		12	12
10											1		11	11	88
9										1		10	10	75	150
8									1		9	9	63	126	471
7								1		8	8	52	104	372	960
6							1		7	7	42	84	287	735	2163
5						1		6	6	33	66	215	546	1566	4262
4					1		5	5	25	50	155	390	1090	2930	8076
3				1		4	4	18	36	106	264	719	1908	5178	14060
2			1		3	3	12	24	67	165	438	1147	3066	8232	22305
1		1		2	2	7	14	37	90	233	602	1586	4212	11299	30536
0	1		1	1	3	6	15	36	91	232	603	1585	4213	11298	30537
y \ x	0	1	2	3	4	5	6	7	8	9	10	11	12	12	14

Table 6: Number of $\{(1,1),(1,-k) : k \in \mathbb{Z}_{>0}\}$-paths that do not cross below the x-axis.

13		1													
12		1	12												
11		1	11	68											
10		1	10	57	252										
9		1	9	47	194	709									
8		1	8	38	146	505	1646								
7		1	7	30	107	350	1092	3313							
6		1	6	23	76	235	702	2057	5960						
5		1	5	17	52	152	435	1232	3472	9765					
4		1	4	12	34	94	258	707	1940	5337	14728				
3		1	3	8	21	55	145	385	1030	2775	7525	20526			
2		1	2	5	12	30	76	196	512	1353	3610	9713	26324		
1		1	1	3	6	15	36	91	232	603	1585	4213	11298	30537	
0	1		1	1	3	6	15	36	91	232	603	1585	4213	11298	30537
y \ x	0	1	2	3	4	5	6	7	8	9	10	11	12	12	14

Table 7: Number of $\{(1,k),(1,-1) : k \in \mathbb{Z}_{>0}\}$-paths that do not cross below the x-axis.

10											1
9										1	10
8									1	9	63
7								1	8	52	312
6							1	7	42	238	1316
5						1	6	33	176	930	4908
4					1	5	25	125	630	3206	16470
3				1	4	18	84	403	1976	9860	49912
2			1	3	12	52	237	1119	5424	26832	134913
1		1	2	7	28	121	550	2591	12536	61921	310954
0	1	1	3	11	45	197	903	4279	20793	103049	518859
y \ x	0	1	2	3	4	5	6	7	8	9	10

Table 8: Number of $\{(1,1),(0,-k) : k \in \mathbb{Z}_{>0}\}$-paths that do not cross below the x-axis.

[2] Nicholas Ham. *Lattice path enumeration algorithms*, September 2018, https://gitlab.com/n-ham-paper-files/lattice-path-algorithms.

12													1
11												1	
10											1		11
9										1		10	
8									1		9		72
7								1		8		60	
6							1		7		49		350
5						1		6		39		266	
4					1		5		30		195		1335
3				1		4		22		136		901	
2			1		3		15		88		564		3825
1		1		2		9		50		310		2056	
0	1		1		4		21		126		818		5594
y \ x	0	1	2	3	4	5	6	7	8	9	10	11	12

Table 9: Number of $\{(1,1),(0,-2k) : k \in \mathbb{Z}_{>0}\}$-paths that do not cross below the x-axis.

15																1
14															1	
13														1		
12													1			13
11												1			12	
10											1			11		
9										1			10			95
8									1			9			81	
7								1			8			68		
6							1			7			56			483
5						1			6			45			374	
4					1			5			35			280		
3				1			4			26			200			1687
2			1			3			18			133			1095	
1		1			2			11			78			627		
0	1			1			5			34			267			2279
y \ x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Table 10: Number of $\{(1,1),(0,-3k) : k \in \mathbb{Z}_{>0}\}$-paths that do not cross below the x-axis.